Some Implications of Pricing Bundles

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October 31, 2012

Abstract

In this paper, we consider a problem in which two suppliers can sell their respective products both individually and together as a bundle, and study the impact of bundle pricing. Four pricing models (centralized, decentralized, coop-comp and comp-coop) are analyzed with regard to the competition formats and sequences. As one would expect, the firms are always better off when pricing decisions are centralized. However, rather surprisingly, we find that firms may be worse off if the bundle prices are set in a cooperative way; we provide analytical characterization of those instances. Numerical studies show that these insights also hold for some non-linear demand.

1. Introduction

Consider two products, 1 and 2, that are characterized by three demand streams—the first two, being direct, are the independent demands for the two products; the third demand is for the bundled product that involves both products 1 and 2. Examples of this type of setting are numerous; for instance, computer components can be purchased individually, or as a kit. Dell, a virtual assembler, sells such kits of products, many of whom can be directly purchased from its suppliers (for example, Sony monitors). Another classic example is air tickets—consider three cities, A, B and C. Tickets can be purchased for itineraries A-B, B-C, and the bundle A-B-C, with the first two possibly being operated by different airlines.

Literatures in economics, transportation, operations, and marketing have studied the pricing and inventorying of such products. In addition, questions on existence of equilibria and welfare implications have been studied. For example, in the operations literature, complementary products produced by competing firms have been analyzed in the context of assembly systems (Wang and Gerchak [13]; Gerchak and Wang [7]; Wang [12]). We, on the other hand, in this note do a complete analysis of the effects of cooperation and competition in a model with two products, say 1 and 2, that can be sold individually or as a bundle, and when demand is linear and deterministic.

There are several papers that are directly or tangentially related to this work. Netessine and Zhang [9] look at the effects of competitively inventorying such products and were possibly the first to introduce such a model to the literature. Fang and Wang [6] consider a model with n partially complementary products in which they assume that the manufacturers' prices for satisfying the common demand exceed the prices for filling their individual demands and show that all the manufacturers first satisfy the common demand, and only then their individual demand. Fang [5] extends the study by allowing the price for common demand to be either higher or lower than the individual demand. Perakis et al. [10] look at a "price of anarchy"-type analysis with complements and point out welfare insights. Armstrong and Vickers [2] consider a model in which both firms offer both products and compete for customers in all markets (for individual products and bundles). Our interest is to study the effects of bundling alone, so we assume that each firm offers a single product.

There is a large literature on bundling and tying of products. The findings from that body of work have implications in various settings. Much of this literature ignore the existence of individual products in the market, or the ramifications of partial cooperation on the pricing of the bundle; our analysis in this paper does this. For a good survey of bundling and welfare ramifications, please see http://www.justice.gov/atr/public/hearings/ip/chapter_5.htm#ii.

In the transportation economics literature, Bilotkach [3] and Czerny [4] are two examples. The latter looks at a very similar model in the context of airline networks. Most recently, Armstrong [1] also studies scenarios in which bundle discount could be set by an integrated firm or two individual firms. While [1] is more relevant to symmetric firms and customer welfare, we in this paper aim to characterize a full picture of the equilibria, and identify areas in which cooperation may harm firm-side profit. As far as we know, the insights we get from our analysis seem to be unreported in the literature.

2. Assumptions

Consider a population of customers with size M and two complementary products indexed 1 and 2. A customer may demand product 1, product 2, both products, or no products at all, with probability γ_{01}, γ_{10}, γ_{11} and γ_{00}, respectively. The potential markets for product 1 and product 2 alone are denoted as $a_1 = M\gamma_{01}$ and $a_2 = M\gamma_{10}$, and the potential market for the bundle is $b = M\gamma_{11}$.

The parameters a_1, a_2, and b, therefore, partially reflect the demand correlation between the two individual products; e.g., large b with small a's implies strong positive correlation, while large a's and small b suggest negative correlation.

Suppose that product i is sold by firm i with individual demand $D_i = a_i - \lambda_i p_i$, $i = 1, 2$; and that the bundle has demand $D_B = b - \lambda^B p^B$. To simplify the analysis, we assume that $\lambda_1 = \lambda_2 = \lambda^B = 1$ and that the cost of each product is zero. These assumptions are made mainly for the ease of presentation and without much loss of generality. In fact, a non-zero cost can be readily incorporated into the model through a slight revision of our demand functions. Similar argument applies to the λ's [1].

Without loss of generality, let us assume $a_1 \leq a_2$, and let denote $p = p_1 + p_2$. We will also use notation $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

3. The Model

We consider three separate cases: the centralized model, in which a single decision maker sets prices that maximize the revenue obtained from sale of all three products; the decentralized model, in which each firm, i, selects prices p_i and p_i^B (so that the bundle price is $p^B = p_1^B + p_2^B$) that maximize its individual profit; and the cooperative cases, in which firm i sets price p_i that maximizes its profit from selling product i, while the price of the bundle is determined cooperatively, so that the profit from the sale of the bundle is maximized.

In each of the above cases, we fully characterize the prices set by the decision makers. The insights we get from analyzing the equilibrium outcomes seem to be interesting. The first insight, that may seem somewhat evident, is that the firms are better off when they cooperate when pricing both the bundle and the individual products. The second insight is surprising and has important welfare implications. When the two firms cooperate on the bundle pricing, depending on the values of the parameters, there are a significant set of instances where their overall profit can be lower than in the competitive case. We analytically characterize when this happens. The direct implication of this fact is that one has to be careful when deciding policies on allowing firms to cooperate when pricing bundles.

We now analyze each of the three cases in detail.

3.1 Centralized Model

We first assume that a single decision maker maximizes the total profit, i.e., sets prices p_1, p_2, and p^B that,

$$\begin{aligned} \max_{p_1, p_2, p^B} \quad & p_1(a_1 - p_1)^+ + p_2(a_2 - p_2)^+ + p^B(b - p^B)^+ \\ s.t. \quad & p_1 + p_2 \geq p^B \geq p_i \quad for \quad i = 1, 2 \\ & p_1, p_2, p^B \geq 0 \end{aligned}$$

where $(\cdot)^+ = \max\{\cdot, 0\}$.

As it is usually the case in practice, we assume that the price of the bundle does not exceed the total of its components (otherwise bundle customers would buy the two items separately), or

[1] The assumptions may affect certain threshold conditions, but would not change the equilibrium structure or the key insights. Although discussions of the impact of cost or demand elasticity on the results could be further expanded, this somewhat deviates from the main focus of this short note, and thus will not be elaborated.

falls below any individual price (in which case customers demanding only product i could buy the bundle and drop product j). The main result for this case is the following.

Proposition 1. (Centralized Solution) *Suppose that a centralized decision maker maximizes the total profit from selling individual products and the bundle. The optimal prices are given as follows:*

Range of the Bundle Market b	p_1	p_2	p^B
$\left[0, (\sqrt{2}-1)a_2\right]$	$\frac{a_1}{2}$	$\frac{a_2}{2}$	-
$\left((\sqrt{2}-1)a_2, a_2\right]$ *and* $b \geq 2a_1 - a_2$	$\frac{a_1}{2}$	$\frac{a_2+b}{4}$	$\frac{a_2+b}{4}$
$\left((\sqrt{2}-1)a_2, a_2\right]$ *and* $b \leq 2a_1 - a_2$	$\frac{a_1+a_2+b}{6}$	$\frac{a_1+a_2+b}{6}$	$\frac{a_1+a_2+b}{6}$
$\left(a_2, a_1+a_2\right]$	$\frac{a_1}{2}$	$\frac{a_2}{2}$	$\frac{b}{2}$
$\left(a_1+a_2, (\sqrt{3}+1)a_1+a_2\right]$	$\frac{b+2a_1-a_2}{6}$	$\frac{b+2a_2-a_1}{6}$	$\frac{2b+a_1+a_2}{6}$
$\left((\sqrt{3}+1)a_1+a_2, \infty\right]$	-	$\frac{a_2}{2}$	$\frac{b}{2}$.

Specifically, (i) if $b \leq a_2$*, then* $p^B = p_2$*; (ii) if* $a_2 < b \leq a_1 + a_2$*, then* $p_2 < p^B = b/2 < p_1 + p_2$*; and (iii) if* $a_1 + a_2 < b$*, then* $p^B = p_1 + p_2$*.*

Note that, when $a_1 + a_2 \geq b$, we have $p \geq p^B$; that is, higher demand for individual products leads to their higher price. On the other hand, $a_1 + a_2 < b$ implies $p = p^B$. In other words, when demand for the bundle is high, individual products are priced the same whether they are sold as a bundle or not. In addition, when $(\sqrt{3}+1)a_1 + a_2 < b$, then $p_1 > a_1$; when demand for an individual product is very low, it is priced out of the market.

3.2 Decentralized Model

In this section, we assume that firm i determines unilaterally p_i and p_i^B that maximize her total profit, which is comprised of the profit from selling individual product, i, and her share of the profit from selling the bundle,

$$p_i(a_i - p_i)^+ + p_i^B(b - p^B)^+,$$

under constraints similar to the ones used in the previous case: $p_j^B \geq p_i - p_i^B \geq p_j^B - p_j$ and $p_i, p_i^B \geq 0$.

Clearly, the profit obtained from the sale of the bundle depends on the price of the other firm as well, hence the price that firm i sets for its share of the bundle is obtained by using the Nash equilibrium (NE). The main result in this case is given by the next proposition.

Proposition 2. (Decentralized Equilibrium) *If each firm maximizes its own profit, the equilibrium prices are given as follows:*

Range of the Bundle Market b	*Equilibria* (p_1, p_2, p_1^B, p_2^B)
$b \leq \frac{3}{4}a_2$,	
• $b < a_1/2$	$\mathbf{p}_a$: *Bundle not offered*
• $\frac{4a_2 - 3a_1}{6} \leq b < \frac{5a_1 - 2a_2}{4}$, $\frac{4+7\sqrt{2}}{20+14\sqrt{2}}a_1 + \frac{4}{20+14\sqrt{2}}a_2 \leq b$	$\mathbf{p}_b$: *Full offerings; Deep discount on bundle*
• $\frac{5a_1 - 2a_2}{4} \leq b$, $\frac{4+5\sqrt{2}}{2(6+5\sqrt{2})}a_2 \leq b$	$\mathbf{p}_c$: *Full offerings; Deep discount on bundle*
$\frac{3}{4}a_2 \leq b \leq \frac{3}{4}(a_1 + a_2)$,	$\mathbf{p}_d$: *Full offerings; Moderate discount on bundle*
$\frac{3}{4}(a_1 + a_2) \leq b \leq \frac{3}{4}(\sqrt{2}+1)(a_1 + a_2)$,	
• $\frac{2a_1 + 6a_2}{7} \leq b \leq \frac{6+5\sqrt{2}}{8}(a_1 + a_2)$	$\mathbf{p}_e$: *Full offerings; No discount on bundle*
• $\frac{3}{4}(a_1 + (\sqrt{2}+1)a_2) < b$	$\mathbf{p}_f$: *Only one product and bundle offered in the market*
• $\frac{3}{4}((\sqrt{2}+1)a_1 + a_2) < b$	$\mathbf{p}_g$: *Only one product and bundle offered in the market*
$\frac{3}{4}(\sqrt{2}+1)(a_1 + a_2) < b$	$\mathbf{p}_f, \mathbf{p}_g$: *Only one product and bundle offered in the market* $\mathbf{p}_h$: *Only bundle offered in the market*

where

$$
\begin{aligned}
\mathbf{p}_a &= (\frac{a_1}{2}, \frac{a_2}{2}, -, -) \\
\mathbf{p}_b &= \left(\frac{a_1+a_2+2b}{7}, \frac{a_1+a_2+2b}{7}, \frac{4a_1-3a_2+b}{7}, \frac{4a_2-3a_1+b}{7}\right) \\
\mathbf{p}_c &= \left(\frac{a_1}{2}, \frac{a_2+2b}{5}, \frac{3b-a_2}{5}, \frac{2a_2-b}{5}\right) \\
\mathbf{p}_d &= (\frac{a_1}{2}, \frac{a_2}{2}, \frac{b}{3}, \frac{b}{3}) \\
\mathbf{p}_e &= \left(p, \frac{a_1+a_2+2b}{5} - p, \frac{4a_1-a_2+3b}{5} - 2p, \frac{2a_2-3a_1-b}{5} + 2p\right), \\
& \quad \textit{where } \frac{a_1}{2} \vee \frac{4a_1-(6+5\sqrt{2})a_2+8b}{20} \le p \le \frac{(2+\sqrt{2})a_1}{4} \wedge \frac{2a_1-3a_2+4b}{10} \\
\mathbf{p}_f &= (p, -, \frac{2a_1+b-4p}{3}, \frac{b-a_1+2p}{3}), \\
& \quad \textit{where } \frac{a_1}{2} \le p \le \frac{(2+\sqrt{2})a_1}{4} \wedge \frac{2a_1-3(\sqrt{2}+1)a_2+4b}{10}. \\
\mathbf{p}_g &= (-, \frac{a_2+2b-3p}{5}, \frac{3b-a_2-2p}{5}, \frac{2a_2-b+4p}{5}), \\
& \quad \textit{where } \frac{(1+\sqrt{2})a_1}{2} \vee \frac{8b-(6+5\sqrt{2})a_2}{12} \le p \le \frac{4b-3a_2}{6}. \\
\mathbf{p}_h &= \left(-, -, \frac{b}{3}, \frac{b}{3}\right)
\end{aligned}
$$

The result suggests that bundle will not be offered in equilibrium when the market for it is small (i.e., $b \le a_1/2$). Conditioned on the bundle being offered, the bundle price p^B can be as low as p_2 when the market is relatively small (i.e., such that $\mathbf{p}_2$ or $\mathbf{p}_3$ is the NE), or as high as the sum of the individual product prices, p, when the market is moderate (i.e., such that $\mathbf{p}_5$ or $\mathbf{p}_8$ is the NE). However, contrary to the results obtained in the centralized case, deeper discount ($p^B < p$) will occur as the bundle market size increases further. This, together with the "bundle-not-being-offered" scenario, reflects how competition can distort equilibrium prices from the first-best outcomes.

3.3 Cooperative Models

Finally, we assume that the two firms jointly determine p^B which maximizes the profit from selling the bundle, $p^B(b - p^B)$, and that each firm, i, selects the price of individual product, p_i, that maximizes her profit, $p_i(a_i - p_i)$. In our analysis, we assume that the firms split the profit from selling the bundle equally among themselves. Note that this allocation of profits can be justified by using several different approaches – it belongs to the core of the corresponding cooperative game, it corresponds to the Shapley value allocations, and to divisions obtained by using most reasonable notions of cooperative bargaining.

We first assume that the firms set the price of the bundle first, and then determine their individual prices. Thus, the model have two stages: the first is cooperative, and the second is competitive. This *coop-comp* model leads to the following result.

Proposition 3. (Coop-Comp Equilibrium) *When the first stage of the model is cooperative, and the second is competitive, the firms make the following decisions in equilibrium:*

Range of the Bundle Market b	p_1	p_2	p^B
$b \leq a_1$	$\frac{b}{2}$	$\frac{b}{2}$	$\frac{b}{2}$
$a_1 < b \leq a_2$	$\frac{a_1}{2}$	$\frac{b}{2}$	$\frac{b}{2}$
$a_2 < b \leq a_1 + a_2$	$\frac{a_1}{2}$	$\frac{a_2}{2}$	$\frac{b}{2}$.
$a_1 + a_2 < b$	$p \in [\frac{a_1}{2}, \frac{b - a_2}{2}]$	$\frac{b}{2} - p$	$\frac{b}{2}$.

Now, suppose that the players first determine their individual prices, and then jointly decide the bundle price. Thus, the model again have two stages: the first is competitive, and the second is cooperative. This *comp-coop* model leads to the following result.

Proposition 4. (Comp-Coop Equilibrium) *When the first stage of the model is competitive, and the second is cooperative, the firms make the following decisions.* [2]

[2] For areas with multiple equilibria, only the one with maximum total profit is shown.

Range of the Bundle Market b	(p_1, p_2, p^B)
$(\sqrt{\frac{3}{2}}+1)b \leq a_2$	$(\frac{a_1}{2}, \frac{a_2}{2}, -)$
$b \leq a_1 \leq a_2 \leq (\sqrt{\frac{3}{2}}+1)b$	$(\frac{2a_2+b}{6}, \frac{2a_2+b}{6}, \frac{2a_2+b}{6})$
$a_1 \leq b \leq a_2 < (\sqrt{\frac{3}{2}}+1)b$	$(\frac{a_1}{2}, \frac{2a_2+b}{6}, \frac{2a_2+b}{6})$
$a_1 \leq a_2 \leq b < a_1 + a_2$	$(\frac{a_1}{2}, \frac{a_2}{2}, \frac{b}{2})$
$a_1 + a_2 \leq b \leq a_1 + a_2 + \frac{4}{\sqrt{3}}a_1$	$(\frac{3a_1 - a_2 + b}{8}, \frac{3a_2 - a_1 + b}{8}, \frac{a_1 + a_2 + b}{4})$
$a_1 + (\sqrt{3}+1)a_2 \leq b$	$(\frac{a_1}{2}, -, \frac{b}{2})$
$(\sqrt{3}+1)a_1 + a_2 \leq b$	$(-, \frac{a_2}{2}, \frac{b}{2})$

For both the *coop-comp* and *comp-coop* models, there is $p^B = p_2 \geq p_1$ when $b \leq a_2$, and $p_2 < p^B < p$ when $b > a_2$. We now further compare the two models in determining the timing sequence that is most beneficial for the firms. It leads to the following result.

Corollary 1. *Suppose that the firms cooperatively determine the price of the bundle.*

1. *For small or large bundle market ($b \leq a_2$ or $b \geq a_1 + a_2$), the firms are jointly better off when they determine their individual prices* before *pricing the bundle.*

2. *For bundle market of moderate size ($a_2 \leq b \leq a_1 + a_2$), the firms generate the same profits under either sequence of pricing.*

The corollary suggests that, overall, *comp-coop* can be a better mechanism than *coop-comp*. Further more, the timing sequence matters only when there is an obvious difference between the individual markets and bundle market. Specifically, a firm should always attend to pricing in its main market first.

4. Comparisons of the models

In this section, we look at the models over the space of parameter values, and compare their effects on the different parties in the market. Should multiple equilibria exist in some region/model, we pick the one that generates maximum total profit for comparison.

We first compare the decentralized model described in Section 3.2 with the centralized model described in Section 3.1.

Proposition 5. (Centralized vs. Decentralized)

- *Both centralized and decentralized models charge the same individual prices (p_i and p_j) when the bundle market is extremely small (i.e., $b \leq (\sqrt{2}-1)a_2$), or large (i.e., $b \geq (\sqrt{3}+1)a_1+a_2$).*
- *Otherwise, decentralized firms charge higher prices than the centralized model for both the bundle (p^B) and for individual products (p_i and p_j).*

Thus, when demand for the bundle is in the extremes, the competing firms charge the same prices as the centralized decision maker. However, once the demand for the bundle is more moderate, competition leads to higher prices.

Next, we compare the cooperative models described in Section 3.3 with the centralized model described in Section 3.1.

Proposition 6. (Centralized vs. Coop-Comp)

- *Both centralized and coop-comp models charge the same prices (individual and bundle) when the bundle market size is moderate (i.e., $a_2 \leq b \leq a_1 + a_2$) or extremely large (e.g., $b \geq (\sqrt{3}+1)a_1 + a_2$).*
- *The coop-comp model charges weakly lower individual prices than the centralized model when the bundle market is not too large (i.e., $b \leq a_1 + a_2$).*
- *The coop-comp model charges lower bundle prices than the centralized model unless the bundle market is moderately large (i.e., $b \geq a_2$).*

Proposition 7. (Centralized vs. Comp-Coop)

- *Both centralized and comp-coop models charge the same prices (individual and bundle) when the bundle market size is moderate (i.e., $a_2 \leq b \leq a_1 + a_2$) or extremely large (e.g., $b \geq (\sqrt{3}+1)a_1 + a_2$).*
- *The comp-coop model charges weakly higher individual and bundle prices than the centralized model when the bundle market is relatively small (i.e., $b \leq a_2$), and weakly lower individual and bundle price when the bundle market is large (i.e., $b > a_1 + a_2$).*

The comparison yields quite contrary results than the one between the decentralized and centralized model. To begin with, both cooperative models arrive at the same equilibrium as the centralized solution under moderate bundle market size. If this does not happen, the cooperative models very often achieve lower individual or even bundle prices (unless the bundle market is large) than the centralized solution. Thus, the distortion brought by the "partial" cooperation is the opposite of that induced by the full competition.

Given the traditional wisdom that cooperation protects its members from internal friction and improves the overall welfare, a natural question to ask is, then, does this "partial" cooperation also leads the firms towards a better position than full competition? We investigate this next, and obtain our main result.

Theorem 1. (Decentralized vs. Cooperative Models) *The effects of the cooperation can be described as follows:*

- *The firms are better off only when the bundle market is sufficiently large (e.g., $b \geq \frac{3}{4}a_2$).*
- *Cooperation, in any sequence, may hurt the profit of the firms when the bundle market is small (e.g., $b < \frac{a_1}{2}$).*

The theorem above shows that the cooperation on bundle pricing can be a double-edged sword. The effects of cooperation depend greatly on the relative sizes of the demands for individual products and their relationship to the demand for the bundle. Proper cooperation will help firms retain the right share in the bundle market, which justifies the impact on the pricing of their individual product. However, when the bundle market is less lucrative, either form of cooperation on bundle pricing will create redundant constraint on individual component pricing, that drives down the overall profit. Therefore, decision about cooperation on bundle product pricing is not straightforward.

4.1 Robustness Test

We acknowledge that Prosition 5, 6, 7, and Theorem 1 were derived on the basis of linear demand functions. Nevertheless, throughout extensive numerical analysis, we find that these insights sustain under certain non-linear demand also. To highlight this fact, consider linear-power demand where $D_i = (a_i - p_i)^\alpha$ and $D_B = (b - p^B)^\alpha$ with $\alpha > 1$. The linear-power demand has been used in a number of marketing and operations works (e.g., Tyagi [11], Yin [14], Huang et al [8]) concerning complementary products, therefore quite relevant to the bundling context. In addition, this demand form does not affect the assumption in §2 or the analysis with the four models. The equilibria can then be derived in a similar fashion. Numerical studies show that the comparative results among the four models generally hold under linear-power demand. We illustrate this through a simple example below:

Example 1. Consider two products with individual market, $a_1 = 1$ and $a_2 = 2$, respectively. The power is $\alpha = 2$. Then the demand for product i in its individual market is $D_i = [max\{0, 1 - p_i\}]^2$, and for the bundle is $D_B = [max\{0, b - p^B\}]^2$. We allow the bundle market size b be taken within $[0, 16]$ with minimum step 0.1. Equilibrium solutions of the four models generate the following observations:

In comparing the centralized model with decentralized model, $({p_1}^c, {p_2}^c) = ({p_1}^d, {p_2}^d)$ when $b \leq 0.8$ or when $b > 8.5$; otherwise, $({p_1}^c, {p_2}^c, p^{B^c}) \leq ({p_1}^d, {p_2}^d, p^{B^d})$. Hence Proposition 5 also holds.

In comparing the centralized model with coop-comp model, we find that $({p_1}^c, {p_2}^c, p^{B^c}) = ({p_1}^{coop-comp}, {p_2}^{coop-comp}, p^{B^{coop-comp}})$ when $b \in [2, 3.2]$ and $b > 5$. $p^{B^c} > p^{B^{coop-comp}}$ when $b < 2$ and $p^{B^c} \leq p^{B^{coop-comp}}$ when $b \geq 3$. When $b \leq 3.2$, there is also ${p_1}^c \geq {p_1}^{coop-comp}$ and ${p_2}^c \geq {p_2}^{coop-comp}$. These are consistent with Proposition 6.

In comparing the centralized model and the comp-coop model, when $b \in [2, 3]$ and $b \geq 4.9$, $({p_1}^c, {p_2}^c, p^{B^c}) = ({p_1}^{comp-coop}, {p_2}^{comp-coop}, p^{B^{comp-coop}})$. Overall, the individual and bundle prices in comp-coop are weakly higher when $b \leq 2$ and weakly lower when $b \geq 3$. These observations conform with Proposition 7.

Finally, compare the profits under centralized and the two cooperative models. We find that $\pi^d \geq \pi^{comp-coop}$ when $b \leq 1$ (in particular, "$<$" holds for $b \in [0.8, 1)$), and $\pi^d > \pi^{coop-comp}$ when $b < 1.5$. When $b \geq 1.5$, both cooperative models yield higher total profit than the decentralized model. This reinforces our findings in Theorem 1 that cooperation might hurt the overall profit of the firms. □

5. Conclusions

Our analysis demonstrates that one cannot reach obvious conclusions on whether cooperation is beneficial when bundling products as compared to competition. The analysis demonstrates that this entirely depends on the relative sizes of the market and their interaction in terms of the parameter values. The firms may be worse off as a result of cooperation; the situations in which this happens are characterized by the bundle market size being below a certain threshold. Moreover, the timing sequence of making a cooperative decision is also important. We believe these findings can be useful for researchers in operations, information economics, and transportation sciences. From our analysis, we believe that the nature of our results will continue to hold in more complicated settings with a network of products with possible bundles of nodes having their own demands.

Acknowledgement

This research is partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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Appendix

The following Lemma will be used in the proofs of propositions.

Lemma 1. *For given $x, y, \lambda > 0$ and $y \geq q \geq 0$, the solution to*

$$\max_p \pi = \max_p p(x-p)^+ + \lambda(p+q)(y-q-p)^+$$
$$s.t. \quad p \geq 0$$

is

- *when $0 < x \leq y-q$,*
  - *(i) if $x-q \leq y-2q < (\sqrt{(1+\lambda)/\lambda}+1)x$, then $p^* = \dfrac{x+\lambda(y-2q)}{2(1+\lambda)}$ and $\pi^* = \dfrac{[x+\lambda(y-2q)]^2}{4(1+\lambda)} + \lambda q(y-q)$;*
  - *(ii) if $(\sqrt{(1+\lambda)/\lambda}+1)x \leq y-2q$, then $p^* = \dfrac{y-2q}{2}$ and $\pi^* = \lambda\dfrac{(y-2q)^2}{4} + \lambda q(y-q)$.*
- *when $0 \leq y-q \leq x$,*
  - *(i) if $y-q \leq x < (\sqrt{1+\lambda}+1)y-2q$, then $p^* = \dfrac{x+\lambda(y-2q)}{2(1+\lambda)}$ and $\pi^* = \dfrac{[x+\lambda(y-2q)]^2}{4(1+\lambda)} + \lambda q(y-q)$;*
  - *(ii) if $(\sqrt{1+\lambda}+1)y-2q \leq x$, then $p^* = \dfrac{x}{2}$ and $\pi^* = \dfrac{x^2}{4}$.*

Proof. For $0 < x \leq y-q$, consider the following two options:

1. Suppose $p \leq x \leq y-q$, then $\pi = -(1+\lambda)p^2 + [x+\lambda(y-2q)]p + \lambda q(y-q)$. The first-order condition (FOC) solution $p^* = \dfrac{x+\lambda(y-2q)}{2(1+\lambda)}$ sustains ($p^* \leq x$) iff $x \leq y-2q \leq (\dfrac{1}{\lambda}+2)x$. Otherwise, $p^* = x$ and $\pi^* = x(y-x)$.

2. Suppose $x < p \leq y-q$, then $\pi = -\lambda p^2 + \lambda(y-2q)p + \lambda q(y-q)$. The FOC solution $p^* = (y-2q)/2$ sustains ($x < p^*$) iff $y-2q > 2x$. Otherwise, $p^* = x$ and $\pi^* = x(y-x)$.

We now compare the above two options. Obviously, when $x \leq y-q < 2x$ there should be $p^* = \dfrac{x+\lambda(y-2q)}{2(1+\lambda)}$, and when $y-q > (\dfrac{1}{\lambda}+2)x$ there should be $p^* = y/2$. When $2x \leq y-q \leq (\dfrac{1}{\lambda}+2)x$, comparing $\pi(\dfrac{x+\lambda(y-2q)}{2(1+\lambda)}) = \dfrac{[x+\lambda(y-2q)]^2}{4(1+\lambda)} + \lambda q(y-q)$ versus $\pi(y/2) = \lambda(y-2q)^2/4 + \lambda q(y-q)$ one can verify that the former is greater when $(\sqrt{\dfrac{1+\lambda}{\lambda}}+1)x > y-2q$.

The case with $0 \leq y-q \leq x$ can be proved in a similar fashion. □

Proof of Proposition 1: The centralized problem is formulated as

$$(\textbf{CP}): \max_{p_1,p_2,p^B} \Pi^c(p_1,p_2,p^B) = \max_{p_1,p_2,p^B} \quad p_1(a_1-p_1)^+ + p_2(a_2-p_2)^+ + p^B(b-p^B)^+$$
$$\text{s.t.} \quad p^B \leq p_1+p_2$$
$$p^B \geq p_1$$
$$p^B \geq p_2.$$

Given (p_1, p_2), the decision for p^B should follow

$$p^B = \begin{cases} 1.b/2, & \text{if } p_1 + p_2 \geq b/2 \text{ , } p_1 < b/2 \text{ and } p_2 < b/2; \\ 2.p_1, & \text{if } p_1 \geq b/2 \text{ and } p_1 \geq p_2; \\ 3.p_2, & \text{if } p_2 \geq b/2 \text{ and } p_1 < p_2; \\ 4.p_1 + p_2, & \text{if } p_1 + p_2 < b/2. \end{cases}$$

We assume that $a_1 \leq a_2$. Then we have the following analysis.

1. Suppose $p^B = b/2$. Then (**CP**) is equivalent to the global maximum of the following three sub-problems:

   (a) $\max_{p_1,p_2} p_1(a_1 - p_1)$ s.t. $p_1 + p_2 \geq b/2, p_1 \leq b/2, p_2 \leq b/2$, and $p_2 \geq a_2$. The stationary point is $p_1^* = a_1/2$ if $b \geq 2a_2$; otherwise, the feasible solution set is empty.

   (b) $\max_{p_1,p_2} p_2(a_2 - p_2)$ s.t. $p_1 + p_2 \geq b/2, p_1 \leq b/2, p_2 \leq b/2$, and $p_1 \geq a_1$. The stationary point is $p_2^* = a_2/2$ if $b \geq \max\{a_2, 2a_1\}$; $p_2^* = b/2$ if $2a_1 \leq b \leq a_2$; otherwise, the feasible solution set is empty.

   (c) $\max_{p_1,p_2} p_1(a_1 - p_1) + p_2(a_2 - p_2)$ s.t. $p_1 + p_2 \geq b/2, p_1 \leq b/2$, and $p_2 \leq b/2$. The stationary points are $(p_1^*, p_2^*) = (a_1/2, a_2/2)$ if $a_2 \leq b \leq a_1 + a_2$; $(p_1^*, p_2^*) = (\frac{b + a_1 - a_2}{4}, \frac{b + a_2 - a_1}{4})$ if $b \leq 3a_1 + a_2$; $(p_1^*, p_2^*) = (\frac{a_1}{2}, \frac{b}{2})$ if $a_1 \leq b \leq a_2$; and $(p_1^*, p_2^*) = (\frac{b}{2}, \frac{b}{2})$ if $b \leq a_1$.

   Comparing the objective values of each problem above, we can determine that

Scenario:	p_1	p_2	p^B	Π^c
$b \leq a_1$	$b/2$	$b/2$	$b/2$	$a_1 b/2 + a_2 b/2 - b^2/4$
$a_1 \leq b \leq a_2$	$a_1/2$	$b/2$	$b/2$	$a_1^2/4 + a_2 b/2$
$a_2 \leq b \leq a_1 + a_2$	$a_1/2$	$a_2/2$	$b/2$	$a_1^2/4 + a_2^2/4 + b^2/4$
$a_1 + a_2 \leq b$	$b/2$	$a_2/2$	$b/2$	$a_2^2/4 + b^2/4.$

2. Suppose now that $p^B = p_1$, and solve $\max_{p_1,p_2} p_1(a_1 - p_1)^+ + p_1(b - p_1)^+ + p_2(a_2 - p_2)$ s.t. $p_1 \geq b/2, p_1 \geq p_2$. First consider $a_1 \leq b \leq (\sqrt{2}+1)a_1$. Note that $(a_1 + b)/4 \leq b/2$. By Lemma 1 the optimal solution is $(p_1^*, p_2^*) = (b/2, a_2/2)$ when $b \geq a_2 \geq a_1$ and $(p_1^*, p_2^*) = (b/2, b/2)$ when $a_1 \leq b \leq a_2$. If $b > (\sqrt{2}+1)a_1$, then the optimal solution is $(p_1^*, p_2^*) = (b/2, a_2/2)$ when $b \geq a_2$ and $(p_1^*, p_2^*) = (b/2, b/2)$ when $(\sqrt{2}+1)a_1 < b \leq a_2$. Now consider $b < a_1 \leq a_2$. If $b < a_1 \leq (\sqrt{2}+1)b$, then the optimal solution is $(p_1^*, p_2^*) = ((b+a_1)/4, a_2/2)$ when $b + a_1 \geq 2a_2$ (does not hold) and $(p_1^*, p_2^*) = ((a_1 + a_2 + b)/6, (a_1 + a_2 + b)/6)$ when $b + a_1 < 2a_2$. If $(\sqrt{2}+1)b < a_1$, then the optimal solution is $(p_1^*, p_2^*) = ((a_1 + a_2 + b)/6, (a_1 + a_2 + b)/6)$.

Scenario:	p_1	p_2	p^B	Π^c
$a_2 \le b$	$b/2$	$a_2/2$	$b/2$	$a_1 b/2 + a_2^2/4$
$a_1 \le b < a_2$	$b/2$	$b/2$	$b/2$	$a_1 b/2 + a_2 b/2 - b^2/4$
$b \le a_1 \le a_2$	$(a_1 + a_2 + b)/6$	$(a_1 + a_2 + b)/6$	$(a_1 + a_2 + b)/6$	$(a_1 + a_2 + b)^2/12$

3. Similarly, for $p^B = p_2$, solve $\max_{p_1,p_2} p_2(a_2 - p_2)^+ + p_2(b - p_2)^+ + p_1(a_1 - p_1)$ s.t. $p_2 \ge b/2, p_2 \ge p_1$. First consider $a_2 \le b \le (\sqrt{2}+1)a_2$. Note that $(a_2 + b)/4 \le b/2$. By Lemma 1 the optimal solution is $(p_1^*, p_2^*) = (a_1/2, b/2)$. If $b > (\sqrt{2}+1)a_2$, then the optimum is still $(p_1^*, p_2^*) = (a_1/2, b/2)$. Now consider $b < a_2$. If $b < a_2 \le (\sqrt{2}+1)b$, then the optimal solution is $(p_1^*, p_2^*) = (a_1/2, (b+a_2)/4)$ when $b \ge 2a_1 - a_2$ and $(p_1^*, p_2^*) = ((a_1+a_2+b)/6, (a_1+a_2+b)/6)$ when $b < 2a_1 - a_2$. If $(\sqrt{2}+1)b < a_2$, then the stationary point is $(p_1^*, p_2^*) = (a_2/2, a_1/2)$.

Scenario:	p_1	p_2	p^B	Π^c
$a_2 \le b$	$a_1/2$	$b/2$	$b/2$	$a_2 b/2 + a_1^2/4$
$b < a_2 < (\sqrt{2}+1)b$ and $b \ge 2a_1 - a_2$	$a_1/2$	$(a_2 + b)/4$	$(a_2 + b)/4$	$(a_2 + b)^2/8 + a_1^2/4$
$b < a_2 < (\sqrt{2}+1)b$ and $b < 2a_1 - a_2$	$(a_1 + a_2 + b)/6$	$(a_1 + a_2 + b)/6$	$(a_1 + a_2 + b)/6$	$(a_1 + a_2 + b)^2/12$
$(\sqrt{2}+1)b \le a_2$	$a_1/2$	$a_2/2$	$a_2/2$	$a_1^2/4 + a_2^2/4$

4. Finally, for $p^B = p_1 + p_2$, we solve:

   (a) $\max_{p_1,p_2} (p_1 + p_2)(b - p_1 - p_2)$ s.t. $p_1 + p_2 \le b/2, p_1 \ge a_1, p_2 \ge a_2$. The stationary point is $p_1^* \in [a_1, b/2 - a_2]$ and $p_2^* = b/2 - p_1^*$ if $a_1 + a_2 \le b/2$. Otherwise, the feasible solution set is empty.

   (b) $\max_{p_1,p_2} p_2(a_2 - p_2) + (p_1 + p_2)(b - p_1 - p_2)$ s.t. $p_1 + p_2 \le b/2, p_1 \ge a_1$. The stationary point is $(p_1^*, p_2^*) = (\frac{b - a_2}{2}, \frac{a_2}{2})$ if $b \ge 2a_1 + a_2$; $(p_1^*, p_2^*) = (a_1, b/2 - a_1)$ if $2a_1 \le b \le 2a_1 + a_2$; otherwise, the feasible set is empty.

   (c) $\max_{p_1,p_2} p_1(a_1 - p_1) + p_2(a_2 - p_2) + (p_1 + p_2)(b - p_1 - p_2)$ s.t. $p_1 + p_2 \le b/2, p_1 \ge 0, p_2 \ge 0$. The stationary points are $(p_1^*, p_2^*) = (\frac{2a_1 - a_2 + b}{6}, \frac{2a_2 - a_1 + b}{6})$ if $b \ge a_1 + a_2$; $(p_1^*, p_2^*) = (\frac{a_1 - a_2 + b}{4}, \frac{a_2 - a_1 + b}{4})$ if $3a_1 + a_2 \ge b \ge a_2 - a_1$; $(p_1^*, p_2^*) = (0, \frac{a_2 + b}{4})$ if $b \ge a_2$; and $(p_1^*, p_2^*) = (0, \frac{b}{2})$ if $b \le a_2$.

   Consequently, we have

Scenario:	p_1	p_2	p^B	Π^c
$b \leq a_2 - a_1$	0	$b/2$	$b/2$	$a_2 b/2$
$a_2 - a_1 \leq b \leq a_1 + a_2$	$\dfrac{a_1 - a_2 + b}{4}$	$\dfrac{a_2 - a_1 + b}{4}$	$b/2$	$\dfrac{a_1^2 + a_2^2 + b^2 - 2a_1a_2 + 2a_1b + 2a_2b}{8}$
$a_1 + a_2 \leq b \leq$ $\leq (\sqrt{3}+1)a_1 + a_2$	$\dfrac{2a_1 - a_2 + b}{6}$	$\dfrac{2a_2 - a_1 + b}{6}$	$\dfrac{a_1 + a_2 + 2b}{6}$	$\dfrac{a_1^2 + a_2^2 + b^2 + a_1b + a_2b - a_1a_2}{6}$
$(\sqrt{3}+1)a_1 + a_2 \leq b$	$\dfrac{b - a_2}{2}$	$\dfrac{a_2}{2}$	$\dfrac{b}{2}$	$\dfrac{a_2^2 + b^2}{4}.$

Now, comparing the four tables that we have so far yields:

Scenario	Optimal Bundle Pricing Decision
$b \leq a_2$	$p^B = p_2$
$a_2 < b \leq a_1 + a_2$	$p^B = b/2$
$a_1 + a_2 < b$	$p^B = p_1 + p_2.$

□

In what follows, we use $A \sim B$ to denote any real number in interval $[A, B]$.

In order to prove Proposition 2, we first need the following result.

Lemma 2. *For given $x, y, w > 0$ and $w \geq u$, the solution to*

$$\begin{aligned} \max_{p,q} \quad & p(x-p)^+ + q(y-q)^+ \\ s.t. \quad & u \leq p - q \leq w \\ & p \geq 0 \\ & q \geq 0 \end{aligned}$$

is:

(i) if $2u \leq x - y \leq 2w$, then $(p^*, q^*) = (\frac{x}{2}, \frac{y}{2})$;

(ii) if $x - (\sqrt{2}+1)y \leq 2w < x - y$, then $(p^*, q^*) = \left(\frac{x+y}{4} + \frac{w}{2}, \frac{x+y}{4} - \frac{w}{2}\right)$;

(iii) if $2w < x - (\sqrt{2}+1)y < x - y$, then $(p^*, q^*) = \left(\frac{x}{2}, \frac{x}{2} - w \sim \frac{x}{2} - u\right)$;

(iv) if $(\sqrt{2}+1)x - y \geq 2u > x - y$, *then* $(p^*, q^*) = \left(\frac{x+y}{4} + \frac{u}{2},\ \frac{x+y}{4} - \frac{u}{2}\right)$*;*

(v) if $x - y \leq (\sqrt{2}+1)x - y < 2u$, *then* $(p^*, q^*) = \left(\frac{y}{2} + u \sim \frac{y}{2} + w,\ \frac{y}{2}\right)$.

Proof. On $\mathbb{R}^{2+}$, define $\mathbb{A} = \{(p,q) : u \leq p - q \leq w\}$, $\mathbb{B} = \{(p,q) : p - q > w\}$ and $\mathbb{C} = \{(p,q) : p - q < u\}$. Clearly $S^o = (x/2, y/2)$ is the optimal solution if it is in $\mathbb{A}$. Otherwise,

- if $S^o \in \mathbb{B}$ and $(x/2 - y/2\sqrt{2}, y/2 + y/2\sqrt{2}) \notin \mathbb{B}$, then the optimal solution is $\left(\frac{x+y}{4} + \frac{w}{2},\ \frac{x+y}{4} - \frac{w}{2}\right)$;
- if $S^o \in \mathbb{B}$ and $(x/2 - y/2\sqrt{2}, y/2 + y/2\sqrt{2}) \in \mathbb{B}$, then the optimal solution is $\left(\frac{x}{2},\ \frac{x}{2} - w \sim \frac{x}{2} - u\right)$;
- if $S^o \in \mathbb{C}$ and $(x/2 + x/2\sqrt{2}, y/2 - x/2\sqrt{2}) \notin \mathbb{C}$, then the optimal solution is $\left(\frac{x+y}{4} + \frac{u}{2},\ \frac{x+y}{4} - \frac{u}{2}\right)$;
- if $S^o \in \mathbb{C}$ and $(x/2 + x/2\sqrt{2}, y/2 - x/2\sqrt{2}) \in \mathbb{C}$, then the optimal solution is $\left(\frac{y}{2} + u \sim \frac{y}{2} + w,\ \frac{y}{2}\right)$; □

Proof of Proposition 2: Firm i needs to determine p_i and p_i^B given p_j and p_j^B:

$$\begin{aligned} \max_{p_i, p_i^B} \quad & p_i(a_i - p_i)^+ + p_i^B(b - p_j^B - p_i^B)^+ \\ \text{s.t.} \quad & p_i + p_j \geq p_i^B + p_j^B \\ & p_i^B + p_j^B \geq p_i \\ & p_i \geq 0 \\ & p_i^B \geq 0 \end{aligned}$$

If some firm announces a $p_j^B \geq b$, it indicates that this firm does not intend to supply the bundle market, thus the only equilibrium shall be $(p_1^*, p_2^*) = (a_1/2, a_2/2)$ and $p^{B*} \geq b$. the optimal response is $p_i^* = a_i/2 \wedge p_j^B \vee (p_j^B - p_j)^+$ and $p_i^B* = 0$. Other than this trivial case, consider $p_j^B < b$. Then, by Lemma 2, the optimal reaction function is

1. *Distinct Full Line*—both individual item i and the bundle B will be offered. The bundle price will be distinct from p_i the price of product i, or the total of product i and j, $p_i + p_j$: $(p_i^*, p_i^{B*}) = \left(\frac{a_i}{2}, \frac{b - p_j^B}{2}\right)$ if $2p_j^B - 2p_j \leq a_i - b + p_j^B \leq 2p_j^B$ and $p_j^B < b$;

2. *Marginal Full Line*—both individual item i and the bundle B will be offered. The bundle price is the same as the firm's individual item, i.e., $p_i = p^B$: $(p_i^*, p_i^{B*}) = \left(\frac{a_i + b + p_j^B}{4}, \frac{a_i + b - 3p_j^B}{4}\right)$ if $a_i - (\sqrt{2}+1)(b - p_j^B) \leq 2p_j^B < a_i - b + p_j^B$ and $p_j^B < b$;

3. *Bundle Free*—bundle product will be priced out of the market, $p^B \geq b$: $(p_i^*, p_i^{B*}) = \left(\frac{a_i}{2}, \frac{a_i}{2} - p_j^B \sim \frac{a_i}{2} - p_j^B + p_j\right)$ if $2p_j^B < a_i - (\sqrt{2}+1)(b - p_j^B) \leq a_i - b + p_j^B$ and $p_j^B < b$;

4. *Discount-Free Full Line*—both individual item i and the bundle B will be offered. Bundle discount does not exist , i.e., $p^B = p_i + p_j$:
   $(p_i^*, p_i^{B*}) = \left(\frac{a_i + b + p_j^B - 2p_j}{4}, \frac{a_i + b - 3p_j^B + 2p_j}{4}\right)$, if $(\sqrt{2}+1)a_i - b + p_j^B \geq 2p_j^B - 2p_j \geq a_i - b + p_j^B$ and $p_j^B \leq b - a_i$;

5. *Product-i Free*—product i will be priced out of the market, i.e., $p_i \geq a_i$:
   $(p_i^*, p_i^{B*}) = \left(\frac{b + p_j^B - 2p_j}{2} \sim \frac{b + p_j^B}{2}, \frac{b - p_j^B}{2}\right)$ if $(\sqrt{2}+1)a_i - b + p_j^B < 2p_j^B - 2p_j$ and $p_j^B \leq b - a_i$.

We next analyze possible equilibria. Denote $\mathbf{p}_{mn}$ as equilibrium pricing when firm 1 uses strategy m (indexed as above) and firm 2 is with strategy n respectively, where $m, n \in \{1, 2, 3, 4, 5\}$. Note that not all equilibria is feasible. Take $\mathbf{p}_{13}$ for example, in equilibrium bundle cannot be offered and priced out of market at the same time. The feasible equilibria are listed below (Italic term in a price vector indicates dummy pricing, i.e., the particular item will be priced out of market, and the price itself is just for technical purpose.).

$$\mathbf{p}_{11} = \left(\frac{a_1}{2}, \frac{a_2}{2}, \frac{b}{3}, \frac{b}{3}\right).$$

Feasible when $\frac{3}{4}a_2 \le b \le \frac{3}{4}(a_1 + a_2)$.

$$\mathbf{p}_{22} = \left(\frac{a_1 + a_2 + 2b}{7}, \frac{a_1 + a_2 + 2b}{7}, \frac{4a_1 - 3a_2 + b}{7}, \frac{4a_2 - 3a_1 + b}{7}\right),$$

Feasible when $\frac{4a_2 - 3a_1}{6} \le b < \frac{5a_1 - 2a_2}{4}$, and $\frac{4a_1 + (4 + 7\sqrt{2})a_2}{20 + 14\sqrt{2}} \le b$.

$$\mathbf{p}_{33} = \left(\frac{a_1}{2}, \frac{a_2}{2}, p_1^B, \frac{a_2}{2} - p_1^B\right), \quad where \quad (\sqrt{2}+1)^2 b - (\sqrt{2}+1)a_1 < p_1^B < a_1 - b.$$

Feasible when $b < \frac{a_1}{2}$.

$$\mathbf{p}_{44} = \left(p_1, \frac{a_1 + a_2 + 2b}{5} - p_1, \frac{4a_1 - a_2 + 3b}{5} - 2p_1, \frac{2a_2 - 3a_1 - b}{5} + 2p_1\right), \quad where \quad \frac{a_1}{2} \le p_1 \le \frac{(2+\sqrt{2})a_1}{4}$$

and $\frac{4a_1 - (6 + 5\sqrt{2})a_2 + 8b}{20} \le p_1 \le \frac{2a_1 - 3a_2 + 4b}{10}$.

Feasible when $\frac{3}{4}(a_1 + a_2) \le \frac{2a_1 + 7a_2}{6} \le b \le \frac{6 + 5\sqrt{2}}{8}(a_1 + a_2)$.

$$\mathbf{p}_{55} = \left(p_1, \frac{2}{3}b - p_1, \frac{b}{3}, \frac{b}{3}\right), \quad where \quad \frac{(\sqrt{2}+1)}{2}a_1 < p_1 < \frac{4b - 3(\sqrt{2}+1)a_2}{6}.$$

Feasible when $b > \frac{3}{4}(\sqrt{2}+1)(a_1 + a_2)$.

$$\mathbf{p}_{12} = \left(\frac{a_1}{2}, \frac{a_2 + 2b}{5}, \frac{3b - a_2}{5}, \frac{2a_2 - b}{5}\right).$$

Feasible when $5a_1 - 2a_2 \le 4b$, and $\frac{4 + 5\sqrt{2}}{2(6 + 5\sqrt{2})}a_2 \le b \le \frac{3}{4}a_2$.

$$\mathbf{p}_{15} = \left(\frac{a_1}{2}, -\frac{a_1}{2} + \frac{2}{3}b, \frac{b}{3}, \frac{b}{3}\right).$$ Feasible when $\frac{3}{4}(a_1 + (1 + \sqrt{2})a_2) < b$.

$$\mathbf{p}_{51} = \left(-\frac{a_2}{2} + \frac{2}{3}b, \frac{a_2}{2}, \frac{b}{3}, \frac{b}{3}\right).$$ Feasible when $\frac{3}{4}((1 + \sqrt{2})a_1 + a_2) < b$.

$$\mathbf{p}_{45} = \left(p_1, \frac{a_1 + 2b - 5p_1}{3}, \frac{2a_1 + b - 4p_1}{3}, \frac{b - a_1 + 2p_1}{3}\right), \quad where \quad \frac{a_1}{2} \le p_1 \le \frac{(2+\sqrt{2})a_1}{4}$$

and $p_1 < \frac{2a_1 - 3(\sqrt{2}+1)a_2 + 4b}{10}$.

Feasible when $\frac{3}{4}(a_1 + (1 + \sqrt{2})a_2) < b$.

$$\mathbf{p}_{54} = \left(p_1, \frac{a_2 + 2b - 3p_1}{5}, \frac{3b - a_2 - 2p_1}{5}, \frac{2a_2 - b + 4p_1}{5}\right), \quad where \quad p_1 \ge \frac{(1+\sqrt{2})a_1}{2}, p_1 \ge \frac{8b - (6 + 5\sqrt{2})a_2}{12}$$

and $p_1 \le \frac{4b - 3a_2}{6}$.

Feasible when $\frac{3}{4}((1 + \sqrt{2})a_1 + a_2) < b$.

Note that $\mathbf{p}_{15}$ and $\mathbf{p}_{51}$ are degenerated cases for $\mathbf{p}_{45}$ and $\mathbf{p}_{54}$ respectively. The equilibria are summarized in the following table:

Range of the Bundle Market b	(p_1, p_2, p_1^B, p_2^B)
$b \le \frac{3}{4}a_2$,	
$b < a_1/2$	$\mathbf{p}_a$
$\frac{4a_2 - 3a_1}{6} \le b < \frac{5a_1 - 2a_2}{4}$, $\frac{4a_1 + (4+7\sqrt{2})a_2}{20 + 14\sqrt{2}} \le b$	$\mathbf{p}_b$
$\frac{5a_1 - 2a_2}{4} \le b$, $\frac{4+5\sqrt{2}}{2(6+5\sqrt{2})}a_2 \le b$	$\mathbf{p}_c$
$\frac{3}{4}a_2 \le b \le \frac{3}{4}(a_1 + a_2)$,	$\mathbf{p}_d$
$\frac{3}{4}(a_1 + a_2) \le b \le \frac{3}{4}(\sqrt{2}+1)(a_1 + a_2)$,	
$\frac{2a_1 + 7a_2}{6} \le b \le \frac{6+5\sqrt{2}}{8}(a_1 + a_2)$	$\mathbf{p}_e$
$\frac{3}{4}(a_1 + (\sqrt{2}+1)a_2) < b$	$\mathbf{p}_f$
$\frac{3}{4}((\sqrt{2}+1)a_1 + a_2) < b$	$\mathbf{p}_g$
$\frac{3}{4}(\sqrt{2}+1)(a_1 + a_2) < b$	$\mathbf{p}_f$
	$\mathbf{p}_g$
	$\mathbf{p}_h$

where

$$
\begin{aligned}
\mathbf{p}_a &= (\frac{a_1}{2}, \frac{a_2}{2}, -, -) \\
\mathbf{p}_b &= \left(\frac{a_1+a_2+2b}{7}, \frac{a_1+a_2+2b}{7}, \frac{4a_1-3a_2+b}{7}, \frac{4a_2-3a_1+b}{7}\right) \\
\mathbf{p}_c &= \left(\frac{a_1}{2}, \frac{a_2+2b}{5}, \frac{3b-a_2}{5}, \frac{2a_2-b}{5}\right) \\
\mathbf{p}_d &= (\frac{a_1}{2}, \frac{a_2}{2}, \frac{b}{3}, \frac{b}{3}) \\
\mathbf{p}_e &= \left(p, \frac{a_1+a_2+2b}{5} - p, \frac{4a_1-a_2+3b}{5} - 2p, \frac{2a_2-3a_1-b}{5} + 2p\right), \\
&\quad \text{where } \frac{a_1}{2} \vee \frac{4a_1-(6+5\sqrt{2})a_2+8b}{20} \leq p \leq \frac{(2+\sqrt{2})a_1}{4} \wedge \frac{2a_1-3a_2+4b}{10} \\
\mathbf{p}_f &= (p, -, \frac{2a_1+b-4p}{3}, \frac{b-a_1+2p}{3}), \\
&\quad \text{where } \frac{a_1}{2} \leq p \leq \frac{(2+\sqrt{2})a_1}{4} \wedge \frac{2a_1-3(\sqrt{2}+1)a_2+4b}{10}. \\
\mathbf{p}_g &= (-, \frac{a_2+2b-3p}{5}, \frac{3b-a_2-2p}{5}, \frac{2a_2-b+4p}{5}), \\
&\quad \text{where } \frac{(1+\sqrt{2})a_1}{2} \vee \frac{8b-(6+5\sqrt{2})a_2}{12} \leq p \leq \frac{4b-3a_2}{6}. \\
\mathbf{p}_h &= \left(-, -, \frac{b}{3}, \frac{b}{3}\right).
\end{aligned}
$$

□

Proof of Proposition 3: Clearly, the bundle price that maximizes the profit is $p^B = b/2$. The decision p_i for firm i, given the price of the other firm p_j, is therefore:

$$
\begin{aligned}
\max_{p_i} \quad & p_i(a_i - p_i)^+ \\
\text{s.t.} \quad & p_i \geq b/2 - p_j \\
& p_i \leq b/2 \\
& p_i \geq 0.
\end{aligned}
$$

The optimal response function is

Scenario:	p_i^*
$p_j \le b/2$,	
$b \le a_i$	$b/2$
$b - 2p_j < a_i < b$	$a_i/2$
$a_i \le b - 2p_j$	$b/2 - p_j$
$p_j > b/2$	
$b \le a_i$	$b/2$
$a_i < b$	$a_i/2$

and the equilibrium is, therefore,

(p_1, p_2)	$b < a_1$	$a_1 \le b$
$b < a_2$	$(b/2, b/2)$	$(a_1/2, b/2)$
$a_2 \le b < a_1 + a_2$	-	$(a_1/2, a_2/2)$
$a_1 + a_2 \le b$	-	$(p, b/2 - p)$ where $p \in [\frac{a_1}{2}, \frac{b-a_2}{2}]$

□

Proof of Proposition 4: We start with the second stage: given the competitive prices p_1 and p_2, the cooperative decision p^B in the second stage is determined by:

$$\begin{aligned} \max_{p^B} \quad & p^B(b - p^B) \\ \text{s.t.} \quad & p^B \le p_1 + p_2 \\ & p^B \ge p_1 \vee p_2 \\ & p^B \ge 0. \end{aligned}$$

The optimal decision should follow:

$$p^{B*} = \begin{cases} b/2, & \text{if } p_1 + p_2 \ge b/2 \text{ , } p_1 < b/2 \text{ and } p_2 < b/2; \\ p_1, & \text{if } p_1 \ge b/2 \text{ and } p_1 \ge p_2; \\ p_2, & \text{if } p_2 \ge b/2 \text{ and } p_1 < p_2; \\ p_1 + p_2, & \text{if } p_1 + p_2 < b/2. \end{cases}$$

Now, we address competitive decision made in the first stage. If firm i receives a half of the bundle profit, her price p_i is determined by:

$$\begin{aligned} (\mathbf{P_i}) : \max_{p_i} \Pi_i = \max_{p_i} \quad & p_i(a_i - p_i)^+ + 0.5p^B(b - p^B)^+ \\ \text{s.t.} \quad & p^B = p^{B*} \\ & p_i \ge 0. \end{aligned}$$

- If $b \le p_j$, obviously $p_i^* = a_i/2$.
- If $p_j < b/2$, first consider $b/2 - p_j \le p_i < b/2$ such that $p^B = b/2$. Then, $p_i^* = (a_i/2 \wedge b/2) \vee (b/2 - p_j)$. Alternatively, for $p_i \ge b/2$ such that $p^B = p_i$, by Lemma 1 the local optimum is $p_i^* = \dfrac{a_i + b/2}{3}$ if $b < a_i \le (\sqrt{\dfrac{3}{2}}+1)b$, $b/2$ if $a_i \le b$, and $\dfrac{a_i}{2}$ if $(\sqrt{\dfrac{3}{2}}+1)b \le a_i$. Finally, for $p_i < b/2 - p_j$ there will be $p^B = p_i + p_j$. Again by Lemma 1 the local optimum is $p_i^* = \dfrac{a_i + (b-2p_j)/2}{3}$ if $\dfrac{b-2p_j}{\sqrt{3}+1} < a_i \le b - 2p_j$, $b/2 - p_j$ if $a_i > b - 2p_j$ or $\dfrac{b-2p_j}{\sqrt{3}+1} \le a_i$. Comparing total profits across the three regions derives the the optimal solution.
- If $b/2 \le p_j < b$, consider $p_i \ge p_j$ such that $p^B = p_i$. Similarly by Lemma 1 for $a_i \le b$, the optimal solution is $p_i = \dfrac{2a_i + b}{6}$ if $a_i \le b \le (\sqrt{3}+1)a_i$ and $a_i/2$ if $(\sqrt{3}+1)a_i \le b$. In either case $p_i \le b/2 \le p_j$ so $p_i^* = p_j$. For $a_i \ge (\sqrt{\dfrac{3}{2}}+1)b$, $p_i^* = a_i/2 \wedge p_j = a_i/2$. For $b \le a_i \le (\sqrt{\dfrac{3}{2}}+1)$, $p_i^* = \dfrac{2a_i+b}{6}$ when $3p_j - b/2 \le a_i$, $p_i^* = p_j$ when $2b \le a_i \le (3p_j - b/2) \wedge [2p_j + \sqrt{2p_j(b-p_j)}]$ or when $a_i \le (3p_j - b/2) \wedge 2b$, $p_i^* = a_i/2$ when $2b \vee [2p_j + \sqrt{2p_j(b-p_j)}] \le a_i \le 3p_j - b/2$. Compare the total profits gives the following table:

Scenario:	p_i^*	p^B	Π_i
$p_j \le b/2$,			
$(\sqrt{\frac{3}{2}}+1)b < a_i$	$a_i/2$	$a_i/2$	$a_i^2/4$
$b \le a_i \le (\sqrt{\frac{3}{2}}+1)b$	$(2a_i+b)/6$	$(2a_i+b)/6$	$(2a_i+b)^2/24$
$b - 2p_j \le a_i \le b$	$a_i/2$	$b/2$	$(2a_i^2+b^2)/8$
$\frac{b-2p_j}{\sqrt{3}+1} < a_i \le b - 2p_j$	$(a_i + b/2 - p_j)/3$	$(a_i + b/2 + 2p_j)/3$	$(a_i + b/2 - p_j)^2/6 + p_j(b-p_j)/2$
$0 \le a_i \le \frac{b-2p_j}{\sqrt{3}+1}$	$b/2 - p_j$	$b/2$	$b^2/8$
$b/2 \le p_j \le b$			
$(\sqrt{\frac{3}{2}}+1)b < a_i$	$a_i/2$	$a_i/2$	$a_i^2/4$
$2p_j \le a_i \le (\sqrt{\frac{3}{2}}+1)b$,			
$3p_j - b/2 \le a_i$	$(2a_i+b)/6$	$(2a_i+b)/6$	$(2a_i+b)^2/24$
$\hat{p}_j \vee 2b \le a_i \le 3p_j - b/2$	$a_i/2$	$a_i/2$	$a_i^2/4$
$2b \le a_i \le (3p_j - b/2) \wedge \hat{p}_j$	p_j	p_j	$p_j(a_i - p_j) + p_j(b-p_j)/2$
$a_i \le (3p_j - b/2) \wedge 2b$	p_j	p_j	$p_j(a_i - p_j) + p_j(b-p_j)/2$
$a_i < 2p_j$	$a_i/2$	p_j	$a_i^2/4 + p_j(b-p_j)/2$
$b \le p_j$	$a_i/2$	p_j	$a_i^2/4$

where $\hat{p}_j = 2p_j + \sqrt{2p_j(b-p_j)}$.

It can be verified that the equilibrium is

(p_1, p_2, p^B)	$(\sqrt{\frac{3}{2}}+1)b < a_1$	$b \le a_1 \le (\sqrt{\frac{3}{2}}+1)b$	$a_1 \le b$
$(\sqrt{\frac{3}{2}}+1)b < a_2$	$(\frac{a_1}{2}, \frac{a_2}{2}, -)$	$(\frac{a_1}{2}, \frac{a_2}{2}, -)$	$(\frac{a_1}{2}, \frac{a_2}{2}, -)$
$b < a_2 \le (\sqrt{\frac{3}{2}}+1)b$	-	(p, p, p) where $p \in [\frac{2a_2+b}{6}, \frac{2a_2+b}{6} + \sqrt{\frac{b^2+4a_2b-2a_2^2}{6}}]$	$(\frac{a_1}{2}, \frac{2a_2+b}{6}, \frac{2a_2+b}{6})$
$a_2 \le b$	-	-	$(\frac{a_1}{2}, \frac{a_2}{2}, \frac{b}{2})$ when $b \le a_1 + a_2$; $(\frac{a_1}{2}, \frac{2a_2-a_1+b}{6}, \frac{2a_1+2a_2+b}{6})$ when $a_1+a_2 \le b \le a_1 + (\sqrt{3}+1)a_2$; $(\frac{2a_1-a_2+b}{6}, \frac{a_2}{2}, \frac{2a_1+2a_2+b}{6})$ when $a_1+a_2 \le b \le (\sqrt{3}+1)a_1 + a_2$; $(\frac{3a_1-a_2+b}{8}, \frac{3a_2-a_1+b}{8}, \frac{a_1+a_2+b}{4})$ when $a_1+a_2 \le b \le a_1+a_2+\frac{4}{\sqrt{3}}a_1$; $(\frac{a_1}{2}, -, \frac{b}{2})$ when $a_1 + (\sqrt{3}+1)a_2 \le b$; $(-, \frac{a_2}{2}, \frac{b}{2})$ when $(\sqrt{3}+1)a_1 + a_2 \le b$; $(-, -, \frac{b}{2})$. when $(1+\sqrt{3})(a_1+a_2) \le b$;

It can be verified that for areas with multiple equilibria, $(\frac{3a_1-a_2+b}{8}, \frac{3a_2-a_1+b}{8}, \frac{a_1+a_2+b}{4})$ yields higher total profit than $(\frac{a_1}{2}, \frac{2a_2-a_1+b}{6}, \frac{2a_1+2a_2+b}{6})$ or $(\frac{2a_1-a_2+b}{6}, \frac{a_2}{2}, \frac{2a_1+2a_2+b}{6})$, $a_1+(\sqrt{3}+1)a_2 \leq b$ $(\frac{b-a_2}{2}, \frac{a_2}{2}, \frac{b}{2})$

□

Proof of Corollary 1: When $a_2 \geq a_1 > b$, the total profit in the cooperative-competitive model equals $\Pi^{coop-comp} = b(2a_1+2a_2-b)/4$ and $\Pi^{comp-coop} = (a_1^2+a_2^2)/4$ when $(\sqrt{3/2}+1)b \leq a_2$ and $\Pi^{comp-coop} = (2a_1+b)(2a_2+b)/12$ when $b \leq a_1 \leq a_2 \leq (\sqrt{3/2}+1)b$. In either case, $\Pi^{comp-coop} \geq \Pi^{coop-comp}$.

When $a_1 \leq b \leq a_2$, $\Pi^{comp-coop} = (a_1^2+a_2 b)/4$ and $\Pi^{comp-coop} = (a_1^2+a_2^2)/4$ when $(\sqrt{3/2}+1)b \leq a_2$ and $\Pi^{comp-coop} = (9a_1^2+4a_2^2+10a_2 b+4b^2)/36$ when $a_1 \leq b \leq a_2 \leq (\sqrt{3/2}+1)b$. In either case, $\Pi^{comp-coop} \geq \Pi^{coop-comp}$.

When $a_2 \leq b \leq a_1+a_2$, $\Pi^{comp-coop} = \Pi^{coop-comp} = (a_1^2+a_2^2+b^2)/4$.

When $a_1+a_2 < b$, $\max_p \Pi^{comp-coop} = (a_1^2+a_2^2-2a_1 b-2a_1 a_2)/4$ when $b < a_2+3a_1$ and $max_p \Pi^{comp-coop} = (a_2^2+b^2)/4$ when $b \geq a_2+3a_1$. $\Pi^{coop-comp} = (5a_1^2+5a_2^2+5b^2-6a_1 a_2+6a_1 b+6a_2 b)/32$ when $b \leq a_2+(1+4/\sqrt{3})a_1$ and $\max \Pi^{coop-comp} = (a_2^2+b^2)/4$. Overall $\Pi^{comp-coop} \geq \Pi^{coop-comp}$.

□

Proof of Proposition 5:
The conclusion can be drawn by simply comparing the pricing decisions in two tables from Proposition 1 and Proposition 2.

□

Proof of Proposition 6:

- If $a_1 \leq (\sqrt{2}-1)a_2$, then the relationship between (p_1^c, p_2^c, p^{Bc}) and $(p_1^{comp-coop}, p_2^{comp-coop}, p^{Bcomp-coop})$ as b increases from $(0, a_1]$, $(a_1, (\sqrt{2}-1)a_2]$, $((\sqrt{2}-1)a_2, a_2]$, $(a_2, a_1+a_2]$ is $(>,>,>)$, $(=,>,>)$, $(=,>,>)$, and $(=,=,=)$ respectively.
- If $(\sqrt{2}-1)a_2 < a_1 \leq a_2$, the relationship between (p_1^c, p_2^c, p^{Bc}) and $(p_1^{comp-coop}, p_2^{comp-coop}, p^{Bcomp-coop})$ as b increases from $(0, (\sqrt{2}-1)a_2]$, $((\sqrt{2}-1)a_2, a_1]$, $(a_1, a_2]$ $(a_2, a_1+a_2]$ is $(>,>,>)$, $(>,>,>)$, $(\geq,>,>)$, and $(=,=,=)$ respectively.

When $b > a_1+a_2$, comparative result can go two way due to multiple equilibria in the coop-comp model. If only the equilibrium with maximum total profit is considered, i.e., $(p_1^{coop-comp}, p_2^{coop-comp}) = (\frac{b-a_2}{2}, \frac{a_2}{2})$, there is still $p_1^c \geq\leq p_1^{coop-comp}$ and $p_2^c \geq p_2^{coop-comp}$.

□

Proof of Proposition 7:

- If $a_1 \leq (\sqrt{6}-2)a_2$, the relationship between (p_1^c, p_2^c, p^{Bc}) and $(p_1^{comp-coop}, p_2^{comp-coop}, p^{Bcomp-coop})$ as b increases from $(0, (\sqrt{2}-1)a_2]$, $((\sqrt{2}-1)a_2, (\sqrt{6}-$

$2)a_2]$, $((\sqrt{6}-2)a_2, a_2]$, $(a_2, a_1+a_2]$, $(a_1+a_2, (\sqrt{3}+1)a_1+a_2]$, $((\sqrt{3}+1)a_1+a_2, \infty]$ is $(=,=,=)$, $(=,<,<)$, $(=,<,<)$, $(=,=,=)$, $(>,>,>)$ and $(=,=,=)$ respectively.

- If $(\sqrt{6}-2)a_2 \leq a_1 \leq a_2$, the relationship between (p_1^c, p_2^c, p^{Bc}) and $(p_1^{comp-coop}, p_2^{comp-coop}, p^{Bcomp-coop})$ as b increases from $(0, (\sqrt{2}-1)a_2]$, $((\sqrt{2}-1)a_2, (\sqrt{6}-2)a_2]$, $((\sqrt{6}-2)a_2, a_1]$ $(a_1, a_2]$, $(a_2, a_1+a_2]$, $(a_1+a_2, (\sqrt{3}+1)a_1+a_2]$, $((\sqrt{3}+1)a_1+a_2, \infty]$ is $(=,=,=)$, $(=,<,<)$, $(<,<,<)$, $(\geq,<,<)$, $(=,=,=)$, $(>,>,>)$ and $(=,=,=)$ respectively.

□

Proof of Theorem 1:
For the decentralized model, firm's profit with respect to each possible equilibrium price is as follows:

Equilibria	π^d
$\mathbf{p}_a$	$\dfrac{a_1^2+a_2^2}{4}$
$\mathbf{p}_b$	$\dfrac{4a_1^2+4a_2^2+2b^2+9a_1b+9a_2b+8a_1a_2}{49}$
$\mathbf{p}_c$	$\dfrac{a_1^2}{4}+\dfrac{3a_2^2+2b^2+7a_2b}{25}$
$\mathbf{p}_d$	$\dfrac{a_1^2+a_2^2}{4}+\dfrac{2b^2}{9}$
$\mathbf{p}_e$	$\dfrac{6a_1^2+6a_2^2+4b^2+14a_1b+14a_2b-13a_1a_2}{50}$
$\mathbf{p}_f$	$\dfrac{a_1^2}{4}+\dfrac{2b^2}{9}$
$\mathbf{p}_g$	$\dfrac{a_2^2}{4}+\dfrac{2b^2}{9}$
$\mathbf{p}_h$	$\dfrac{2}{9}b^2$

For the coop-comp model,

	$\pi^{coop-comp}$
$b \le a_1$	$\frac{a_1 b + a_2 b}{2} - \frac{b^2}{4}$
$a_1 < b \le a_2$	$\frac{a_1^2}{4} + \frac{a_2 b}{2}$
$a_2 < b \le a_1 + a_2$	$\frac{a_1^2 + a_2^2 + b^2}{4}$
$a_1 + a_2 < b$	$p(a_1 - p)^+ + (b/2 - p)(a_2 - b/2 + p)^+ + \frac{b^2}{4} \le \frac{a_2^2 + b^2}{4}$

For the comp-coop model,

	$\pi^{comp-coop}$
$b \le (\sqrt{6} - 2)a_2$	$\frac{a_1^2 + a_2^2}{4}$
$(\sqrt{6} - 2)a_2 \le b \le a_1 \le a_2$	$\frac{(2a_1 + b)(2a_2 + b)}{12}$
$a_1 \le b \le a_2$, $(\sqrt{6} - 2)a_2 \le b$	$\frac{9a_i^2 + 4a_j^2 + 4b^2 + 10a_j b}{36}$
$a_1 \le a_2 \le b < a_1 + a_2$	$\frac{a_1^2 + a_2^2 + b^2}{4}$
$a_1 + a_2 \le b \le a_1 + a_2 + \frac{4}{\sqrt{3}} a_1$	$\frac{5a_i^2 - 6a_i a_j + 5a_j^2 + 6a_i b + 6a_j b + 5b^2}{32}$
$a_1 + (\sqrt{3} + 1)a_2 \le b$	$\frac{a_1^2 + b^2}{4}$
$(\sqrt{3} + 1)a_1 + a_2 \le b$	$\frac{a_2^2 + b^2}{4}$

We note that multiple equilibria can exist when $b < a_1$; if this occurs, we pick the one that yields maximum joint profit for the firms (i.e., $p = a_1/2$).

Now, compare the profits in decentralized model with those in the coop-comp model:

Range of the Bundle Market b	π^d vs. $\pi^{coop-comp}$
$b \leq \frac{3}{4}a_2$,	
$b < a_1/2$	$>$
$\frac{4a_2 - 3a_1}{6} \leq b < \frac{5a_1 - 2a_2}{4}$, $\frac{4a_1 + (4 + 7\sqrt{2})a_2}{20 + 14\sqrt{2}} \leq b$	$>$
$\frac{5a_1 - 2a_2}{4} \leq b$, $\frac{4 + 5\sqrt{2}}{2(6 + 5\sqrt{2})}a_2 \leq b$	$>$
$\frac{3}{4}a_2 \leq b \leq \frac{3}{4}(a_1 + a_2)$,	$<$
$\frac{3}{4}(a_1 + a_2) \leq b \leq \frac{3}{4}(\sqrt{2} + 1)(a_1 + a_2)$,	
$\frac{2a_1 + 7a_2}{6}b \leq \frac{6 + 5\sqrt{2}}{8}(a_1 + a_2)$	$<$
$b\frac{3}{4}(a_1 + (\sqrt{2} + 1)a_2) < b$	
$\frac{3}{4}((\sqrt{2} + 1)a_1 + a_2) < b$	
$\frac{3}{4}(\sqrt{2} + 1)(a_1 + a_2) < b$	$<$
	$<$
	$<$

Finally, compare the profits in decentralized model with those in the comp-coop model:

Range of the Bundle Market b	π^d vs. $\pi^{comp-coop}$
$b \leq \frac{3}{4}a_2$,	
$b < a_1/2$	$><$
$\frac{4a_2 - 3a_1}{6} \leq b < \frac{5a_1 - 2a_2}{4}$, $\frac{4a_1 + (4+7\sqrt{2})a_2}{20+14\sqrt{2}} \leq b$	$><$
$\frac{5a_1 - 2a_2}{4} \leq b$, $\frac{4+5\sqrt{2}}{2(6+5\sqrt{2})}a_2 \leq b$	$<$
$\frac{3}{4}a_2 \leq b \leq \frac{3}{4}(a_1 + a_2)$,	$<$
$\frac{3}{4}(a_1 + a_2) \leq b \leq \frac{3}{4}(\sqrt{2}+1)(a_1 + a_2)$,	
$\frac{2a_1 + 7a_2}{6}b \leq \frac{6+5\sqrt{2}}{8}(a_1 + a_2)$	$<$
$b\frac{3}{4}(a_1 + (\sqrt{2}+1)a_2) < b$	
$\frac{3}{4}((\sqrt{2}+1)a_1 + a_2) < b$	
$\frac{3}{4}(\sqrt{2}+1)(a_1 + a_2) < b$	$<$
	$<$
	$<$

Specifically, $\pi^d > \pi^{comp-coop}$ if $(\sqrt{6} - 2)a_2 < b < \frac{a_1}{2}$.

□